APPLICATIONS OF THE SPACE DIFFERENTIAL GEOMETRY AT THE STUDY OF PRODUCTION FUNCTIONS

Catalin Angelo IOAN, Lecturer, Phd.
"DANUBIUS"University of Galati

Abstract:

This paper is a new onset about production functions. Because all papers on this subject use the projections of production functions on a plan, the analysis becomes heavy and less general in conclusions, and for this reason we made a treatment from the point of view of differential geometry in space.

On the other hand, we generalise the Cobb-Douglas, CES and Sato production functions to a unique form and we made the analysis on this.

The conclusions of the paper allude to the principal directions of the surface (represented by the graph of the production function) i.e. the directions in which the function varies the best. Also the concept of the total curvature of a surface is applied here and we obtain that it is null in every point, that is all points are parabolic.

We compute also the surface element which is useful to finding all production (by means the integral) when both labour and capital are variable

Key words: production function, differential geometry, curvature, principal direction, Cobb-Douglas

JEL Classification: C65

1. INTRODUCERE

Fie funcţia de producţie Q=Q(K,L) unde:

- Q=producţia;
- K=capitalul;
- L=munca

Funcţia $Q:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ trebuie să satisfacă condiţiile:

1. Q(0,0)=0;
2. Q este diferenţiabilă de ordinul 2 în orice punct interior al domeniului de definiţie;
3. $\frac{\partial Q}{\partial K}\geq 0, \frac{\partial Q}{\partial L}\geq 0$;
4. $\frac{\partial^2 Q}{\partial K^2}\leq 0, \frac{\partial^2 Q}{\partial L^2}\leq 0$
5. Q este o funcţie omogenă de grad 1, adică Q(tK,tL)=tQ(K,L) $\forall t\in\mathbf{R}$

Semnificaţia primei condiţii este aceea că la anularea unui factor, producţia devine nulă.

A doua condiţie este utilă numai din considerente pur matematice.

Condiţia a treia semnifică faptul că o creştere a unuia dintre factori (munca sau capitalul) producţia creşte, de asemenea.

1. INTRODUCTION

Let a production function Q=Q(K,L) where:

- Q=product;
- K=capital;
- L=labour

The function $Q:\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ must satisfiy the conditions:

6. Q(0,0)=0;
7. Q is differentiable of order 2 in any interior point of the production set;
8. $\frac{\partial Q}{\partial K}\geq 0, \frac{\partial Q}{\partial L}\geq 0$;
9. $\frac{\partial^2 Q}{\partial K^2}\leq 0, \frac{\partial^2 Q}{\partial L^2}\leq 0$
10. Q is a homogenous function of degree 1, that is Q(tK,tL)=tQ(K,L) $\forall t\in\mathbf{R}$

The meaning of the first condition is that at a vanishing of one factor the product is null.

The second condition is useful just for mathematical calculus.

The third means that at an increase of one factor (labour or capital) the product also grow.

Cea de-a patra condiţie, deoarece derivata a doua reprezintă viteza de variaţie a primeia, semnifică faptul că producia are o viteză mai mică atunci când unul din factori rămâne constant, iarcelălalt variază.

Reprezentarea grafică a unei funcţii de producţie este o suprafaţă.

Fie deci:

$$p=\frac{\partial Q}{\partial L},\ q=\frac{\partial Q}{\partial K},\ r=\frac{\partial^2 Q}{\partial L^2},\ s=\frac{\partial^2 Q}{\partial L\partial K},\ t=\frac{\partial^2 Q}{\partial K^2}.$$

Pentru o valoare constantă a unuia dintre parametri, obţinem o curbă pe o suprafaţă. De exemplu: $Q=Q(K,L_0)$ sau $Q=Q(K_0,L)$ sunt amândouă curbe ale suprafeţei de producţie.

Ele sunt obţinute din intersecţia planuli $L=L_0$ sau $K=K_0$ cu suprafaţa $Q=Q(K,L)$.

Curbura unei curbe este, dintr-un punctelementar de vedere, gradul de deviaţie al acesteia de la o linie dreaptă.

În studiul suprafeţelor, două forme pătratice sunt de un deosebit folos:

Prima formă fundamentală a suprafeţei este:

$g=EdL^2+2FdLdK+GdK^2$

unde:

- $E=1+p^2$;
- $F=pq$;
- $G=1+q^2$.

Elementul de arie este: $d\sigma=\sqrt{EG-F^2}\,dKdL$, iar cel de suprafaţă A unde $(K,L)\in R$ (o regiune din planul K-O-L) este $A=\iint_R d\sigma dKdL$.

Forma a doua fundamentală a suprafeţei este:

$h=\lambda dL^2+2\mu dLdK+\nu dK^2$

unde:

- $\lambda=\frac{r}{\sqrt{1+p^2+q^2}}$;
- $\mu=\frac{s}{\sqrt{1+p^2+q^2}}$;
- $\nu=\frac{t}{\sqrt{1+p^2+q^2}}$.

Considerând cantitatea $\delta=\lambda\nu-\mu^2$ avem:

- Dacă $\delta>0$ în fiecare punct al suprafeţei, vom spune că aceasta este eliptică. Astfel de suprafeţe sunt: hiperboloidul cu două pânze, paraboloidul eliptic şi elipsoidul;

The fourth, because the second derivative is the speed of variation of the first, means that the product has a slower speed when one factor becomes constant and the other varies.

The graph representation of a production function is a surface.

Let:

$$p=\frac{\partial Q}{\partial L},\ q=\frac{\partial Q}{\partial K},\ r=\frac{\partial^2 Q}{\partial L^2},\ s=\frac{\partial^2 Q}{\partial L\partial K},\ t=\frac{\partial^2 Q}{\partial K^2}.$$

For a constant value of one parameter we obtain a curve on the surface. For exemple: $Q=Q(K,L_0)$ or $Q=Q(K_0,L)$ are both curves on the production surface. They are obtained from the intersection of the plane $L=L_0$ or $K=K_0$ with the surface $Q=Q(K,L)$.

The curvature of a curve is from an elementary point of view the degree of deviation of the curve relative to a straight line.

In the study of the surfaces, two quadratic forms are very useful.

The first fundamental quadratic form of the surface is:

$g=EdL^2+2FdLdK+GdK^2$

where:

- $E=1+p^2$;
- $F=pq$;
- $G=1+q^2$.

The area element is $d\sigma=\sqrt{EG-F^2}\,dKdL$ and the surface area A when $(K,L)\in R$ (a region in the plane K-O-L) is $A=\iint_R d\sigma dKdL$.

The second fundamental form of the surface is:

$h=\lambda dL^2+2\mu dLdK+\nu dK^2$

where:

- $\lambda=\frac{r}{\sqrt{1+p^2+q^2}}$;
- $\mu=\frac{s}{\sqrt{1+p^2+q^2}}$;
- $\nu=\frac{t}{\sqrt{1+p^2+q^2}}$.

Considering the quantity $\delta=\lambda\nu-\mu^2$ we have that:

- If $\delta>0$ in each point of the surface, we will say that it is eliptical. Such surfaces are the hyperboloid with two sheets, the eliptical paraboloid and the elypsoid.

- Dacă $\delta<0$ în fiecare punct al suprafeţei, vom spune că aceasta este hiperbolică. Astfel de suprafeţe sunt: hiperboloidul cu o pânză şi hiperboloidul parabolic;
- Dacă $\delta=0$ în fiecare punct al suprafeţei, vom spune că aceasta este parabolică. Astfel de suprafeţe sunt cele conice şi cele cilindrice.

Considerând o suprafaţă S şi o curbă arbitrară printr-un punct P al suprafeţei ce are vectorul tangent v în P, fie planul π determnat de vectorul v şi nrmala N în P la S.Intersecţia lui π cu S este o curbă C_n numită secţiunea normală a lui S. Curbura acesteia se num eşte curbură normală.

- If $\delta<0$ in each point of the surface, we will say that it is hyperbolic. Such surfaces are the the hyperbolid with one sheet and the hyperbolic paraboloid.
- If $\delta=0$ in each point of the surface, we will say that it is parabolic. Such surfaces are the cone surfaces and the cylinder surfaces.

Considering a surface S and an arbitrary curve through a point P of the surface who has the tangent vector v in P, let the pl ane π determined by the vector v and the normal N in P at S. The intersection of π with S is a curve C_n named normal section of S. Its curvature is called normal curvature.

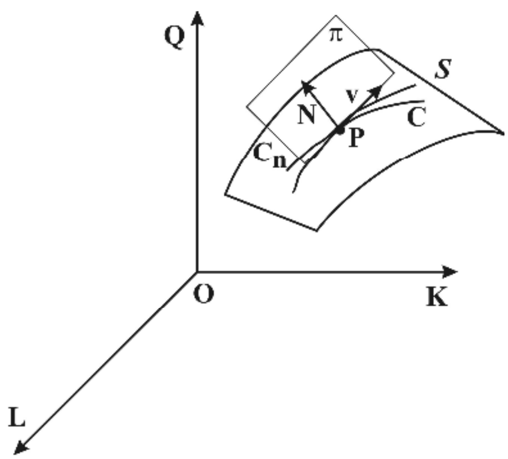

Figure-1: The normal section of a curve

Dacă avem o direcţie $m=\frac{dL}{dK}$ în planul tangent la suprafaţă în P, rezultă că formula curburii normale este dată prin:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}$$

Valorile extreme k_1 şi k_2 ale funcţiei k(m) se numesc curburi principale ale suprafeţei în acel punct. Ele satisfac, de asemenea, ecuaţia:

$$(EG-F^2)k^2-(E\nu-2F\mu+G\lambda)k+(\lambda\nu-\mu^2)=0$$

Valorile lui m ce furnizează aceste extreme se nmesc direcţii principale în acel punct.

Ele satisfac, de asemenea, ecuaţia:

$$(E\mu-F\lambda)m^2+(E\nu-G\lambda)m+(F\nu-G\mu)=0$$

If we have a direction $m=\frac{dL}{dK}$ in the tangent plane of the surface in an arbitrary point P we have that the normal curvature is given by:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}$$

The extreme values k_1 and k_2 of the function k(m) call the principal curvatures of the surface in that point. They satisfy also the equation:

$$(EG-F^2)k^2-(E\nu-2F\mu+G\lambda)k+(\lambda\nu-\mu^2)=0$$

The values of m who give the extremes call principal directions in that point.

They also satisfy the equation:

$$(E\mu-F\lambda)m^2+(E\nu-G\lambda)m+(F\nu-G\mu)=0$$

sau

$(Es-Fr)m^2+(Et-Gr)m+(Ft-Gs)=0$

Curba $\frac{dL}{dK}=m$ (unde m este una din direcţiile principale) se numeşte linie de curbură a suprafeţei. Pe o astfel de curbă avem maximul sau minimul variaţiei lui Q într-o vecinătate a lui P.

Cantitatea $K=k_1k_2$ se numeşte curbură totală în punctul considerat, iar $H=\frac{k_1+k_2}{2}$ se numeşte curbura medie a suprafeţei în acel punct.

Avem, de asemenea;

$K=\frac{\lambda\nu-\mu^2}{EG-F^2}$ şi $H=\frac{E\nu-2F\mu+G\lambda}{EG-F^2}$

O suprafaţă cu K=constant se numeşte suprafaţă de curbură totală constantă, iar dacă H=0 vom spune că avem o surafaţă minimală.

Considerând acum planul tangent π la suprafaţă într-un punct P şi o direcţie m, dacă $\lambda m^2+2\mu m+\nu=0$ vom spune că m este direcţie asimptotică, iar ecuaţia: $\lambda\left(\frac{dL}{dK}\right)^2+2\mu\frac{dL}{dK}+\nu=0$ va furniza curbele asimptotice ale suprafeţei în punctul P.

2. FUNCŢIA GENERALĂ DE PRODUCŢIE

Fie funcţia de producţie:

$Q=A\frac{K^\alpha L^\beta}{(\gamma K^\rho+\varepsilon L^\rho)^\omega}$, $\alpha,\beta,\rho\in[0,1]$, $\omega\in\mathbf{R}$, $\varepsilon+\gamma\neq 0$

- Pentru $\omega=0$, γ,ε,ρ=arbitrari, $\alpha,\beta\in[0,1]$ avem funcţia Cobb-Douglas: $Q=AK^\alpha L^\beta$;
- Pentru $\alpha=0$, $\beta=0$, $\omega=-\frac{1}{\rho}$ avem funcţia CES: $Q=A(\gamma K^\rho+\varepsilon L^\rho)^{\frac{1}{\rho}}$;
- Pentru $\alpha=2$, $\beta=2$, $\rho=3$ şi $\omega=1$ avem funcţia SATO: $Q=A\frac{K^2L^2}{\gamma K^3+\varepsilon L^3}$.
- Pentru a avea o funcţie omogenă de grad 1, avem că: $Q(tK,tL)=tQ(K,L)$ $\forall t\in\mathbf{R}$

Avem, de asemenea:

$Q(tK,tL)=A\,t^{\alpha+\beta-\rho\omega}\frac{K^\alpha L^\beta}{(\gamma K^\rho+\varepsilon L^\rho)^\omega}=t^{\alpha+\beta-\rho\omega}Q(K,L)$

$\Rightarrow \alpha+\beta-\rho\omega=1$.

or

$(Es-Fr)m^2+(Et-Gr)m+(Ft-Gs)=0$

The curve $\frac{dL}{dK}=m$ (where m is one of the principal directions) is called line of curvature on the surface. On such a curve we have the maximum or minimum variation of the value of Q in a neighbourhood of P.

The quantity $K=k_1k_2$ is named the total curvature in the considered point and $H=\frac{k_1+k_2}{2}$ is named the mean curvature of the surface in that point.

We have therefore:

$K=\frac{\lambda\nu-\mu^2}{EG-F^2}$ and $H=\frac{E\nu-2F\mu+G\lambda}{EG-F^2}$

A surface with K=constant call surface with constant total curvature and if H=0 call minimal surface.

Considering now in the tangent plane π at the surface in a point P a direction m, if $\lambda m^2+2\mu m+\nu=0$ we will say that m is an asymptotic direction, and the equation: $\lambda\left(\frac{dL}{dK}\right)^2+2\mu\frac{dL}{dK}+\nu=0$ gives the asymptotic curves of the surface in the point P.

2. THE GENERAL PRODUCTION FUNCTION

Let the production function:

$Q=A\frac{K^\alpha L^\beta}{(\gamma K^\rho+\varepsilon L^\rho)^\omega}$, $\alpha,\beta,\rho\in[0,1]$, $\omega\in\mathbf{R}$, $\varepsilon+\gamma\neq 0$

- For $\omega=0$, γ,ε,ρ=arbitrary, $\alpha,\beta\in[0,1]$ we have the Cobb-Douglas function: $Q=AK^\alpha L^\beta$;
- For $\alpha=0$, $\beta=0$, $\omega=-\frac{1}{\rho}$ we have the CES function: $Q=A(\gamma K^\rho+\varepsilon L^\rho)^{\frac{1}{\rho}}$;
- For $\alpha=2$, $\beta=2$, $\rho=3$ and $\omega=1$ we have the SATO function: $Q=A\frac{K^2L^2}{\gamma K^3+\varepsilon L^3}$.
- In order to have a homogenous function of degree 1, we have that: $Q(tK,tL)=tQ(K,L)$ $\forall t\in\mathbf{R}$

We have therefore:

$Q(tK,tL)=A\,t^{\alpha+\beta-\rho\omega}\frac{K^\alpha L^\beta}{(\gamma K^\rho+\varepsilon L^\rho)^\omega}=t^{\alpha+\beta-\rho\omega}Q(K,L)$

$\Rightarrow \alpha+\beta-\rho\omega=1$.

În consecinţă: $\omega=\frac{\alpha+\beta-1}{\rho}$ şi expresia generală a lui Q va fi:

$$Q=A\frac{K^{\alpha}L^{\beta}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\frac{\alpha+\beta-1}{\rho}}},\ \alpha,\beta,\rho\in[0,1],\ \varepsilon+\gamma\neq 0$$

Avem acum:

$$\frac{\partial Q}{\partial K}=AK^{\alpha-1}L^{\beta}\frac{\gamma(\alpha-\rho\omega)K^{\rho}+\alpha\varepsilon L^{\rho}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega+1}}.$$

Deoarece $(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega}=\frac{AK^{\alpha}L^{\beta}}{Q}$ obţinem:

$$q=\frac{\partial Q}{\partial K}=Q\frac{(1-\beta)\gamma K^{\rho}+\alpha\varepsilon L^{\rho}}{K(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

Prin analogie:

$$p=\frac{\partial Q}{\partial L}=Q\frac{(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}}{L(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

Din relaţiile de mai sus avem acum:

$$t=\frac{\partial^2 Q}{\partial K^2}=-\frac{Q}{K^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$r=\frac{\partial^2 Q}{\partial L^2}=-\frac{Q}{L^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$s=\frac{\partial^2 Q}{\partial K\partial L}=\frac{Q}{KL(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

Să notăm acum:

$P=\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}$

$U=(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}$

$V=\alpha\varepsilon L^{\rho}+(1-\beta)\gamma K^{\rho}$

de unde:

$U+V=\varepsilon L^{\rho}+\gamma K^{\rho}$.

Dacă $\alpha+\beta-1\neq 0$ avem:

$$K^{\rho}=\frac{(1-\alpha)V-\alpha U}{(1-\alpha-\beta)\gamma}\ \text{ şi }\ L^{\rho}=\frac{(1-\beta)U-\beta V}{(1-\alpha-\beta)\varepsilon}.$$

Avem acum:

$$p=\frac{\partial Q}{\partial L}=\frac{QU}{L(U+V)};$$

$$q=\frac{\partial Q}{\partial K}=\frac{QV}{K(U+V)};$$

$$E=1+p^2=1+Q^2\frac{U^2}{L^2(U+V)^2};$$

$$F=pq=Q^2\frac{UV}{KL(U+V)^2};$$

In consequence: $\omega=\frac{\alpha+\beta-1}{\rho}$ and the general expression of Q will be:

$$Q=A\frac{K^{\alpha}L^{\beta}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\frac{\alpha+\beta-1}{\rho}}},\ \alpha,\beta,\rho\in[0,1],\ \varepsilon+\gamma\neq 0$$

We have now:

$$\frac{\partial Q}{\partial K}=AK^{\alpha-1}L^{\beta}\frac{\gamma(\alpha-\rho\omega)K^{\rho}+\alpha\varepsilon L^{\rho}}{(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega+1}}.$$

Because $(\gamma K^{\rho}+\varepsilon L^{\rho})^{\omega}=\frac{AK^{\alpha}L^{\beta}}{Q}$ we obtain:

$$q=\frac{\partial Q}{\partial K}=Q\frac{(1-\beta)\gamma K^{\rho}+\alpha\varepsilon L^{\rho}}{K(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

Through analogy:

$$p=\frac{\partial Q}{\partial L}=Q\frac{(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}}{L(\gamma K^{\rho}+\varepsilon L^{\rho})}$$

With the upper relations we have now:

$$t=\frac{\partial^2 Q}{\partial K^2}=-\frac{Q}{K^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$r=\frac{\partial^2 Q}{\partial L^2}=-\frac{Q}{L^2(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

$$s=\frac{\partial^2 Q}{\partial K\partial L}=\frac{Q}{KL(\gamma K^{\rho}+\varepsilon L^{\rho})^2}[\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}]$$

Let note now:

$P=\alpha(1-\alpha)\varepsilon^2L^{2\rho}+\varepsilon\gamma[(\rho-1)(\alpha+\beta-1)+2\alpha\beta]K^{\rho}L^{\rho}+\beta(1-\beta)\gamma^2K^{2\rho}$

$U=(1-\alpha)\varepsilon L^{\rho}+\beta\gamma K^{\rho}$

$V=\alpha\varepsilon L^{\rho}+(1-\beta)\gamma K^{\rho}$

from where:

$U+V=\varepsilon L^{\rho}+\gamma K^{\rho}$.

If $\alpha+\beta-1\neq 0$ we have:

$$K^{\rho}=\frac{(1-\alpha)V-\alpha U}{(1-\alpha-\beta)\gamma}\ \text{ and }\ L^{\rho}=\frac{(1-\beta)U-\beta V}{(1-\alpha-\beta)\varepsilon}.$$

We have now:

$$p=\frac{\partial Q}{\partial L}=\frac{QU}{L(U+V)};$$

$$q=\frac{\partial Q}{\partial K}=\frac{QV}{K(U+V)};$$

$$E=1+p^2=1+Q^2\frac{U^2}{L^2(U+V)^2};$$

$$F=pq=Q^2\frac{UV}{KL(U+V)^2};$$

$G=1+q^2=1+Q^2\frac{V^2}{K^2(U+V)^2}$.

Cu $\Delta=1+p2+q2=1+Q2\,\frac{K^2U^2+L^2V^2}{K^2L^2(U+V)^2}$

avem:

$\lambda=\frac{r}{\sqrt{\Delta}}$, $\mu=\frac{s}{\sqrt{\Delta}}$, $\nu=\frac{t}{\sqrt{\Delta}}$.

$t=-\frac{QP}{K^2(U+V)^2}$, $r=-\frac{QP}{L^2(U+V)^2}$,

$s=\frac{QP}{KL(U+V)^2}$.

După calcule simple, avem $EG-F^2=1+\frac{Q^2(K^2U^2+L^2V^2)}{K^2L^2(U+V)^2}$

de unde:

$$d\sigma=\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

şi suprafaţa va fi calculată prin:

$$A=\iint_R\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

Direcţiile principale vor fi date de:

$K^2[L^2(U+V)^2+Q^2U^2+Q^2UV]m^2+KL[-L^2(U+V)^2-Q^2U^2+K^2(U+V)^2+Q^2V^2]m-L^2[K^2(U+V)^2+Q^2V^2+Q^2UV]=0$

de unde:

$m_1=\frac{L}{K}$, $m_2=-\frac{L}{K}\,\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}$.

Pentru o direcţie m avem:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}=\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}\cdot\frac{-K^2m^2+2KLm-L^2}{K^2[L^2(U+V)^2+Q^2U^2]m^2+2KLQ^2UVm+L^2[K^2(U+V)^2+Q^2V^2]}$$

Pentru $m_1=\frac{L}{K}$ avem că $k_1=k(m_1)=0$ şi pentru $m_2=-\frac{L}{K}\,\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}$ avem $k_2=k(m_2)=$

$$-\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}.$$

$G=1+q^2=1+Q^2\frac{V^2}{K^2(U+V)^2}$.

With $\Delta=1+p2+q2=1+Q2\,\frac{K^2U^2+L^2V^2}{K^2L^2(U+V)^2}$

we have:

$\lambda=\frac{r}{\sqrt{\Delta}}$, $\mu=\frac{s}{\sqrt{\Delta}}$, $\nu=\frac{t}{\sqrt{\Delta}}$.

$t=-\frac{QP}{K^2(U+V)^2}$, $r=-\frac{QP}{L^2(U+V)^2}$,

$s=\frac{QP}{KL(U+V)^2}$.

After an easy computing we have $EG-F^2=1+\frac{Q^2(K^2U^2+L^2V^2)}{K^2L^2(U+V)^2}$

from where:

$$d\sigma=\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

and the surface area will be compute by:

$$A=\iint_R\frac{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}{KL(U+V)}dKdL$$

The principal directions will be given by:

$K^2[L^2(U+V)^2+Q^2U^2+Q^2UV]m^2+KL[-L^2(U+V)^2-Q^2U^2+K^2(U+V)^2+Q^2V^2]m-L^2[K^2(U+V)^2+Q^2V^2+Q^2UV]=0$

from where:

$m_1=\frac{L}{K}$, $m_2=-\frac{L}{K}\,\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}$.

For a direction m we have:

$$k(m)=\frac{\lambda m^2+2\mu m+\nu}{Em^2+2Fm+G}=\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}\cdot\frac{-K^2m^2+2KLm-L^2}{K^2[L^2(U+V)^2+Q^2U^2]m^2+2KLQ^2UVm+L^2[K^2(U+V)^2+Q^2V^2]}$$

For $m_1=\frac{L}{K}$ we have that $k_1=k(m_1)=0$ and for $m_2=-\frac{L}{K}\,\frac{(U+V)K^2+VQ^2}{(U+V)L^2+UQ^2}$ we have $k_2=k(m_2)=$

$$-\frac{QPLK(U+V)}{\sqrt{K^2L^2(U+V)^2+Q^2(K^2U^2+L^2V^2)}}.$$

$$\frac{[(U+V)K^2+VQ^2]^2+[(U+V)K^2+VQ^2][(U+V)L^2+UQ^2]+[(U+V)L^2+UQ^2]^2}{[L^2(U+V)^2+Q^2U^2][(U+V)K^2+VQ^2]^2-2Q^2UV[(U+V)K^2+VQ^2][(U+V)L^2+UQ^2]+[K^2(U+V)^2+Q^2V^2][(U+V)L^2+UQ^2]^2}$$

Curbura totală a suprafeţei este: $K=k_1k_2=0$.
Curbura medie este, de asemenea:

$$H=\frac{E\nu-2F\mu+G\lambda}{2(EG-F^2)}=-\frac{KLQP(U+V)\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2(U+V)^2+Q^2[K^2U^2+L^2V^2]}^3}$$

Obţinem că suprafaţa de producţie este cu curbură totală nulă, dar u este minimală în niciun punct.

Linia e curbură este:

$$(E\mu-F\lambda)\left(\frac{dL}{dK}\right)^2+(E\nu-G\lambda)\left(\frac{dL}{dK}\right)+(F\nu-G\mu)=0$$

Ca mai sus, obţinem uşor că:

$\frac{dL}{dK}=\frac{L}{K}$ că este: $\frac{dL}{L}=\frac{dK}{K}$ $\Rightarrow$ $L=CK$ cu $C\in(0,\infty)$

respectiv:

$$\frac{dL}{dK}=-\frac{L}{K}\,\frac{(\varepsilon L^\rho+\gamma K^\rho)K^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+(\alpha\varepsilon L^\rho+(1-\beta)\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}{(\varepsilon L^\rho+\gamma K^\rho)L^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+((1-\alpha)\varepsilon L^\rho+\beta\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}$$

Direcţiile asimptotice satisfac:

$\lambda m^2+2\mu m+\nu=0$

adică:

$rm^2+2sm+t=0$

de unde:

$-K^2m^2+2KLm-L^2=0$ deci $m_1=m_2=\frac{L}{K}$.

Curbele asimptotice au ecuaţia:

$\frac{dL}{dK}=m$ (cu m – direcţie asimptotică) deci ele sunt: $L=CK$ cu $C\in(0,\infty)$.

3. APLICAŢII PENTRU FUNCŢIA COBB-DOUGLAS

Pentru funcţia de producţie Cobb-Douglas, adică pentru $\alpha+\beta=1$, $\gamma=1$, $\varepsilon=0$, $\rho=1$ avem:

$U=\beta K$
$V=\alpha K$
$U+V=K$
$P=\alpha\beta K^2$

The total curvature of the surface is $K=k_1k_2=0$.
The mean curvature is also:

$$H=\frac{E\nu-2F\mu+G\lambda}{2(EG-F^2)}=-\frac{KLQP(U+V)\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2(U+V)^2+Q^2[K^2U^2+L^2V^2]}^3}$$

We obtain that the production surface is with null total curvature but it is not minimal in any point.

The line of curvature equation is:

$$(E\mu-F\lambda)\left(\frac{dL}{dK}\right)^2+(E\nu-G\lambda)\left(\frac{dL}{dK}\right)+(F\nu-G\mu)=0$$

Like at upper, we obtain easy that:

$\frac{dL}{dK}=\frac{L}{K}$ that is: $\frac{dL}{L}=\frac{dK}{K}$ $\Rightarrow$ $L=CK$ with $C\in(0,\infty)$

respectively:

$$\frac{dL}{dK}=-\frac{L}{K}\,\frac{(\varepsilon L^\rho+\gamma K^\rho)K^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+(\alpha\varepsilon L^\rho+(1-\beta)\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}{(\varepsilon L^\rho+\gamma K^\rho)L^2(\gamma K^\rho+\varepsilon L^\rho)^{\frac{2(\alpha+\beta-1)}{\rho}}+((1-\alpha)\varepsilon L^\rho+\beta\gamma K^\rho)A^2K^{2\alpha}L^{2\beta}}$$

The asymptotic directions satisfy:

$\lambda m^2+2\mu m+\nu=0$

that is:

$rm^2+2sm+t=0$

from where:

$-K^2m^2+2KLm-L^2=0$ therefore $m_1=m_2=\frac{L}{K}$.

The asymptotic curves have the equation:

$\frac{dL}{dK}=m$ (with m asymptotic direction) therefore they are: $L=CK$ with $C\in(0,\infty)$.

3. APPLICATIONS FOR THE COBB-DOUGLAS FUNCTION

For the Cobb-Douglas production function, that is for $\alpha+\beta=1$, $\gamma=1$, $\varepsilon=0$, $\rho=1$ we have:

$U=\beta K$
$V=\alpha K$
$U+V=K$
$P=\alpha\beta K^2$

$m_1=\frac{L}{K}$, $m_2=-\frac{L}{K}\,\frac{K^3+\alpha K A^2 K^{2\alpha} L^{2\beta}}{KL^2+\beta K A^2 K^{2\alpha} L^{2\beta}}=$

$-\frac{K+\alpha A^2 K^{2\alpha-1} L^{2\beta}}{L+\beta A^2 K^{2\alpha} L^{2\beta-1}}$

şi notând cu $g=\frac{K}{L}$ obţinem:

$m_1=\frac{1}{g}$, $m_2=-g\frac{1+\alpha A^2 g^{-2\beta}}{1+\beta A^2 g^{2\alpha}}$.

$k_1=0$

$k_2=-\frac{Q\alpha\beta KL}{\sqrt{K^2L^2+Q^2(\beta^2K^2+\alpha^2L^2)}}\cdot$

$$\frac{[K^2+\alpha Q^2]^2+[K^2+\alpha Q^2][L^2+\beta Q^2]+[L^2+\beta Q^2]^2}{[L^2+Q^2\beta^2][K^2+\alpha Q^2]^2-2Q^2\alpha\beta[K^2+\alpha Q^2][L^2+\beta Q^2]+[K^2+Q^2\alpha^2][L^2+\beta Q^2]^2}$$

Curbura totală a suprafeţei este: $K=k_1k_2=0$ şi curbura medie:

$H=-\frac{\alpha\beta LQ\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2+Q^2[\beta^2K^2+\alpha^2L^2]}^3}$.

$m_1=\frac{L}{K}$, $m_2=-\frac{L}{K}\,\frac{K^3+\alpha K A^2 K^{2\alpha} L^{2\beta}}{KL^2+\beta K A^2 K^{2\alpha} L^{2\beta}}=-$

$\frac{K+\alpha A^2 K^{2\alpha-1} L^{2\beta}}{L+\beta A^2 K^{2\alpha} L^{2\beta-1}}$

and denoting with $g=\frac{K}{L}$ the endowment with capital we obtain:

$m_1=\frac{1}{g}$, $m_2=-g\frac{1+\alpha A^2 g^{-2\beta}}{1+\beta A^2 g^{2\alpha}}$.

$k_1=0$

$k_2=-\frac{Q\alpha\beta KL}{\sqrt{K^2L^2+Q^2(\beta^2K^2+\alpha^2L^2)}}\cdot$

$$\frac{[K^2+\alpha Q^2]^2+[K^2+\alpha Q^2][L^2+\beta Q^2]+[L^2+\beta Q^2]^2}{[L^2+Q^2\beta^2][K^2+\alpha Q^2]^2-2Q^2\alpha\beta[K^2+\alpha Q^2][L^2+\beta Q^2]+[K^2+Q^2\alpha^2][L^2+\beta Q^2]^2}$$

The total curvature of the surface is $K=k_1k_2=0$ and the mean curvature is:

$H=-\frac{\alpha\beta LQ\left(L^2+K^2+Q^2\right)}{2\sqrt{K^2L^2+Q^2[\beta^2K^2+\alpha^2L^2]}^3}$.

Selective bibliography:

1. Ioan Catalin Angelo (2004), Applied mathematics in economy, EDP, Bucharest
2. Nerlove M. (1965) Estimation and Identification of Cobb -Douglas Production Functions. Skokie, Ill: Rand-McNally.
3. Nerlove M. (1967) "Recent Empirical Studies of the CES and Related Production Functions", in M. Brown, editor, Theory and Empirical Analysis of Production. New York: Columbia University Press.